

Analytical Model of Tidal Distortion and Dissipation for a Giant Planet with a Viscoelastic Core

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ABSTRACT

We present analytical expressions for the tidal Love numbers of a giant planet with a solid core and a fluid envelope. We model the core as a uniform, incompressible, elastic solid, and the envelope as a non-viscous fluid satisfying the $n = 1$ polytropic equation of state. We discuss how the Love numbers depend on the size, density, and shear modulus of the core. We then model the core as a viscoelastic Maxwell solid and compute the tidal dissipation rate in the planet as characterized by the imaginary part of the Love number k_2 . Our results improve upon existing calculations based on planetary models with a solid core and a uniform ($n = 0$) envelope. Our analytical expressions for the Love numbers can be applied to study tidal distortion and viscoelastic dissipation of giant planets with solid cores of various rheological properties, and our general method can be extended to study tidal distortion/dissipation of super-earths.

Key words: planets and satellites: gaseous planets – planets and satellites: interiors

1 INTRODUCTION

Tidal effects play an important role in understanding many puzzles associated with planet/exoplanet formation and evolution. One example involves high-eccentricity migration of giant planets: tidal dissipation in the planet is responsible for circularizing the planet’s orbit, leading to the creation of hot Jupiters (e.g. Wu & Murray 2003, Fabrycky & Tremaine 2007, Correia et al. 2011, Naoz et al. 2012, Storch et al. 2014, Petrovich 2015).

The levels of tidal dissipation in giant planets suggested by both Solar System (e.g. Goldreich & Soter 1966, Yoder & Peale 1981, Lainey et al. 2009, 2012) and extrasolar (e.g. Socrates et al. 2012, Storch & Lai 2014) constraints cannot easily be explained by simple viscous dissipation in the turbulent fluid envelope (Goldreich & Nicholson 1977). While several mechanisms based on wave excitation and dissipation in the envelope (ocean) have been studied (e.g. Ogilvie & Lin 2004, Ivanov & Papaloizou 2007; see Ogilvie 2014 for a review), it remains unclear whether they can provide sufficient dissipation.

Dissipation in the solid cores of giants planets is another possible source of dissipation. Previous works (Remus et al. 2012, Storch & Lai 2014, Remus et al. 2015) have employed analytical formulae for the tidal dissipation in a two-layer planet consisting of a uniform, incompressible, viscoelastic core and a uniform non-dissipative ocean (e.g. Der-

mott 1979). These works demonstrate that, while there are significant uncertainties in the rheologies of the solid core, it is in principle possible for dissipation in the core to be substantial enough to account for existing constraints, particularly the dependence of dissipation on the tidal forcing frequency (Storch & Lai 2014).

The advantage of analytical models for the tidal deformation lies in the use of the “correspondence principle”, in which the analytical formulae derived for the tidal deformation of an elastic body may be generalized to a *viscoelastic* body via introduction of a complex shear modulus (Biot 1954). This allows various rheologies for the solid core to be employed in calculating the tidal dissipation. In this paper, we extend previous works by considering a fluid envelope (ocean) of non-uniform density, rather than a uniform one. In particular, we show that if the ocean obeys the $n = 1$ polytropic equation of state ($P \propto \rho^2$), relatively simple analytical expressions for the tidal Love numbers can be obtained. The $n = 1$ polytrope is appropriate for giant planets, as it correctly reproduces the fact that their radii are nearly independent of their masses.

In section 2, we set up the analytical problem of calculating the tidal distortion in a two-layer giant planet. In section 3, we present the solution for the tidal radial deformation of the core, characterized by the Love number h_{2c} , and the change in the self-gravity of the planet, characterized by the Love number k_2 . In section 4 we give several examples of the uses of these formulae. We discuss our results and conclude in section 5.

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2 SETUP AND SCHEMATIC SOLUTION

We consider distortion of a planet by an $l = 2$ tidal potential. Let the perturber have mass M_* and be a distance a away. Let the planet have mass M_p and radius R , and possess a solid core of radius R_c . We assume the core is incompressible, with uniform density ρ_c and shear modulus μ . We model the planet's fluid envelope as an $n = 1$ polytrope, such that its pressure (P) and density (ρ) profiles satisfy the relation

$$P(r) = K\rho(r)^2, \quad (1)$$

where K is a constant.

2.1 Equilibrium Structure

In the absence of a perturber, the planet is in hydrostatic equilibrium. The planet's gravitational potential Φ and pressure profile P satisfy the equations

$$\nabla^2 \Phi = 4\pi G\rho, \quad (2)$$

$$\nabla P = -\rho \nabla \Phi. \quad (3)$$

It follows that the density profile in the fluid ocean is given by

$$\rho(r) = \rho_0 \frac{\sin[q(1 - r/R)]}{qr/R}, \quad (4)$$

where

$$q^2 = \frac{2\pi G R^2}{K}, \quad (5)$$

$$\rho_0 = \frac{q^2 M_p}{4\pi R^3}. \quad (6)$$

For clarity, we define ρ_{\max} to be the fluid density at $r = R_c +$ (just outside R_c):

$$\rho_{\max} \equiv \rho_0 \frac{\sin q(1 - R_c/R)}{qR_c/R}. \quad (7)$$

Since we demand the planet to be of mass M_p and radius R , and the core to have radius R_c , this leads to a constraint on the core-to-fluid density jump:

$$\frac{\rho_c}{\rho_{\max}} = \frac{3R^2}{q^2 R_c^2} \left[\frac{qR_c}{R} \cot q(1 - R_c/R) + 1 \right]. \quad (8)$$

In practice, for given M_p , R , ρ_c and R_c (or M_c), we solve for q from equations (6)-(8). For completeness, we give the potential Φ inside the planet:

$$\Phi(r) = \begin{cases} \frac{2}{3}\pi G \rho_c (r^2 - R_c^2) - \frac{GM_p}{R} \left[\frac{\sin q(1 - R_c/R)}{qR_c/R} + 1 \right] & (r \leq R_c) \\ -\frac{GM_p}{R} \left[\frac{\sin q(1 - r/R)}{qr/R} + 1 \right] & (R_c \leq r \leq R). \end{cases} \quad (9)$$

2.2 Tidal Perturbation

We now turn on the $l = 2$ tidal perturbation and calculate the resulting deformation of the planet. We set up the problem such that the z axis joins the centers of the planet and the perturber. In this way, the problem has azimuthal symmetry. The perturbing tidal potential (assumed small) is given by

$$\bar{U}(r, \theta) \equiv U(r)Y_{20}(\theta) = -\sqrt{\frac{4\pi}{5}} \frac{GM_*}{a^3} r^2 Y_{20}(\theta), \quad (10)$$

Hence we can assume all perturbed quantities are proportional to Y_{20} . The perturbed Poisson's equation is given by

$$\nabla^2 \bar{\delta\Phi} = 4\pi G \bar{\delta\rho}, \quad (11)$$

where $\bar{\delta X} \equiv \delta X(r)Y_{20}(\theta)$ indicates the Eulerian perturbation to the variable X , and the perturbed equation of hydrostatic equilibrium in the liquid layer of the planet is given by

$$\nabla \bar{\delta P} = -\bar{\delta\rho} \nabla \Phi - \rho \nabla \bar{V}, \quad (12)$$

where $\bar{V} \equiv V(r)Y_{20}(\theta) \equiv \bar{\delta\Phi} + \bar{U}$, and $\rho(r)$ and $\Phi(r)$ are the unperturbed density profile and unperturbed gravitational potential as derived in the previous subsection. The transverse component of Eq. (12) is $\bar{\delta P} = -\rho \bar{V}$, while the radial component is $(\bar{\delta P})' = -\bar{\delta\rho}\Phi' - \rho \bar{V}'$, where $'$ stands for $\partial/\partial r$. These imply that $\bar{\delta\rho} = (\rho'/\Phi') \bar{V}$.

Inside the core, and outside the planet, the perturbed Poisson equation reduces to $\nabla^2 \bar{\delta\Phi} = 0$, and is easily solved, yielding $\delta\Phi = b_1 r^2$ inside the core and $\delta\Phi = b_4 r^{-3}$ outside the planet, with b_1 and b_4 as yet unknown constants. In the fluid envelope, the Poisson equation reduces to

$$\nabla^2 \bar{V} = 4\pi G \bar{\delta\rho} = 4\pi G \frac{\rho'}{\Phi'} \bar{V}. \quad (13)$$

So far the equations in this subsection are general (valid for all envelope equation of state). for the $n = 1$ EOS, $P' = -\rho\Phi'$ gives $(\rho'/\Phi') = -1/(2K)$ and we have

$$\nabla^2 \bar{V} = -\left(\frac{q}{R}\right)^2 \bar{V}. \quad (14)$$

This equation admits a standard solution in the form of spherical Bessel functions (j_2 and y_2):

$$V(r) = b_2 j_2\left(q \frac{r}{R}\right) + b_3 y_2\left(q \frac{r}{R}\right) \quad (15)$$

inside the fluid envelope, with b_2 and b_3 constants to be determined.

The unknown constants b_1, b_2, b_3, b_4 may now be solved for by matching boundary conditions at $r = R$ and at $r = R_c$:

$$\delta\Phi(R_+) = \delta\Phi(R_-), \quad (16)$$

$$\delta\Phi(R_{c,+}) = \delta\Phi(R_{c,-}), \quad (17)$$

$$(\delta\Phi')_{R_-} = (\delta\Phi')_{R_+} = -\frac{3}{R} \delta\Phi(R_+), \quad (18)$$

$$(\delta\Phi' + 4\pi G \rho \xi_r)_{R_{c,-}} = (\delta\Phi' + 4\pi G \rho \xi_r)_{R_{c,+}}. \quad (19)$$

Since the fluid density vanishes at surface of planet, these introduce only one additional unknown: the radial displacement at the core-fluid interface, $\xi_r(R_c)$. Determining $\xi_r(R_c)$ requires solving for the deformation of the core, matching the radial and transverse tractions across the core-fluid interface. The procedure to follow is similar to Love's classic solution for the deformation of an incompressible, uniform, self-gravitating elastic body under an external potential (Love 1911, Greff-Lefftz et al. 2005), with the addition of an external pressure force due to the fluid envelope. We find the final boundary condition is given by

$$\begin{aligned} \frac{19}{5} \mu \frac{\xi_r(R_c)}{R_c} &= -\left(\frac{dP}{dr} + \rho_c g_c\right) \xi_r \\ &+ (\rho_{\max} - \rho_c)(b_1 R_c^2 + U), \end{aligned} \quad (20)$$

where $g_c \equiv (4/3)\pi G \rho_c R_c$ is the gravitational acceleration at

$r = R_c$, and all the quantities on the RHS are evaluated at $r = R_{c,+}$. Note that equation (20) is valid for any envelope EOS. Together, the five boundary conditions, Eqs. (16) – (20), can be solved for all the unknowns.

3 ANALYTICAL LOVE NUMBER FORMULAE

3.1 Non-dissipative tide

We are interested in two dimensionless Love numbers. The first is the tidal Love number of the planet, defined as

$$k_2 \equiv \frac{\delta\Phi(R)}{U(R)}. \quad (21)$$

This specifies the magnitude of the quadrupole potential produced by the distorted planet, $\overline{\delta\Phi} = k_2 U(R) (R/r)^3 Y_{20}(\theta)$ (for $r > R$), and therefore determines the effect of tidal distortion on the planet's orbit. The second is the radial displacement Love number of the core, defined by

$$h_{2c} \equiv -\frac{\xi_r(R_c)g_c}{U(R_c)}. \quad (22)$$

This specifies the shape of the inner core under the combined influences of the external tidal field and the loading due to the fluid envelope.

Following the schematic procedure outlined in the previous section, we find

$$h_{2c} = \frac{5}{q^2} \left(\frac{R}{R_c} \right)^3 \left\{ \alpha \left[1 + \frac{2\bar{\mu}}{5 \left(1 - \frac{\rho_{\max}}{\rho_c} \right)} \right] - 3\lambda \left(1 - \frac{\rho_{\max}}{\rho_c} \right) \right\}^{-1}, \quad (23)$$

$$k_2 = \frac{3h_{2c}}{q\alpha} \left(\frac{R_c}{R} \right)^2 \left(1 - \frac{\rho_{\max}}{\rho_c} \right) + \frac{5\gamma}{q\alpha} - 1, \quad (24)$$

where

$$\bar{\mu} \equiv 19\mu/(2\rho_c g_c R_c), \quad (25)$$

and

$$\begin{aligned} \alpha &= j_1(q) [\chi_c y_1(\chi_c) - 5y_2(\chi_c)] - y_1(q) [\chi_c j_1(\chi_c) - 5j_2(\chi_c)], \\ \lambda &= y_1(q) j_2(\chi_c) - j_1(q) y_2(\chi_c), \\ \gamma &= j_2(q) [\chi_c y_1(\chi_c) - 5y_2(\chi_c)] - y_2(q) [\chi_c j_1(\chi_c) - 5j_2(\chi_c)], \end{aligned} \quad (26)$$

with $\chi_c \equiv qR_c/R$.

3.2 Dissipative tide

We now consider the effects of viscous dissipation in the solid core. According to the correspondence principle (Biot 1954), we may generalize the calculation of tidal distortion of a non-dissipative elastic core by adopting a complex shear modulus μ , where the imaginary part of μ accounts for dissipation in the viscoelastic core. In general, the complex μ depends on the tidal forcing frequency ω in the rest frame of the planet, and its actual form depends on the rheology of the solid (e.g. Henning et al. 2009, Remus et al. 2012). Thus the complex $k_2 = k_2(\omega)$ also depends on the forcing frequency.

For example, assume the perturber is in a circular orbit

with orbital frequency Ω and the planet is spinning with frequency Ω_s . The forcing frequency is then $\omega = 2\Omega - 2\Omega_s$. The torque on the planet and the energy transfer rate from the orbit to the planet due to dissipation may be calculated as (Storch & Lai 2014)

$$T_z = \frac{3}{2} T_0 \text{Im}[k_2(2\Omega - 2\Omega_s)], \quad (27)$$

$$\dot{E} = \frac{3}{2} T_0 \Omega \text{Im}[k_2(2\Omega - 2\Omega_s)], \quad (28)$$

where $T_0 \equiv G(M_*/a^3)^2 R^5$. See Storch & Lai (2014) for the more general case of a perturber on an eccentric orbit.

Note that \dot{E} includes contributions both from dissipation into heat, which occurs solely in the viscoelastic core, and from the torque T_z that acts to synchronize the rotation rate of the planet with the orbital frequency of the perturber. Thus, the true tidal heating rate received by the core is given by

$$\dot{E}_{\text{heat}} = \dot{E} - \Omega_s T_z. \quad (29)$$

4 APPLICATIONS OF LOVE NUMBER FORMULAE

In this section we present several sample applications of the formulae derived in the previous section. First we consider planets with non-dissipative cores, then generalize to a complex shear modulus and compute the tidal dissipation.

4.1 Non-dissipative elastic core

In Figure 1 we present the Love numbers for a giant planet (mass $M_p = M_J$, radius $R = R_J$) with a core of constant density, $\rho_c = 6 \text{ g cm}^{-3}$, but varying size, for several values of the core shear modulus μ . Our nominal reference value for μ is that of undamaged rocky material at Earth-like pressures and temperatures, $\mu_0 = 900 \text{ kbar}$. Damaged rocky, or icy material can have a lower shear modulus $\sim 40 \text{ kbar}$ (Goldsby & Kohlstedt 2001, Henning et al. 2009). However, little is known about both the composition of giant cores and the behavior of rocky/icy materials under high pressures and thus the value of μ is largely a free parameter.

Based on Fig. 1, we note that the Love numbers generally behave as expected. Cores with higher shear moduli are harder to deform, resulting in smaller Love numbers. Planets with cores of larger radii have more mass concentrated in the center, and thus k_2 decreases as a function of R_c . At $R_c = 0$, i.e. in the absence of a core, k_2 correctly defaults to the standard value for an $n = 1$ envelope, $(15/\pi^2) - 1$. At $R_c/R \approx 0.6$, the core mass is equal to planet mass, i.e. the envelope has zero mass but still artificially extends to $R = R_J$. In this case k_2 and h_{2c} default to values for a bare core:

$$h_{2c,0} = \frac{5}{2(1+\bar{\mu})}, \quad k_{2,0} = \frac{3}{2(1+\bar{\mu})} \left(\frac{R_c}{R} \right)^5. \quad (30)$$

In Figure 2 we present the Love numbers for a planet with a core of constant mass, $M_c = 5M_\oplus$, as a function of the core shear modulus μ , for three different core radii. For the range of core sizes considered (up to $R_c/R = 0.15$), changing the shear modulus by 4 orders of magnitude apparently

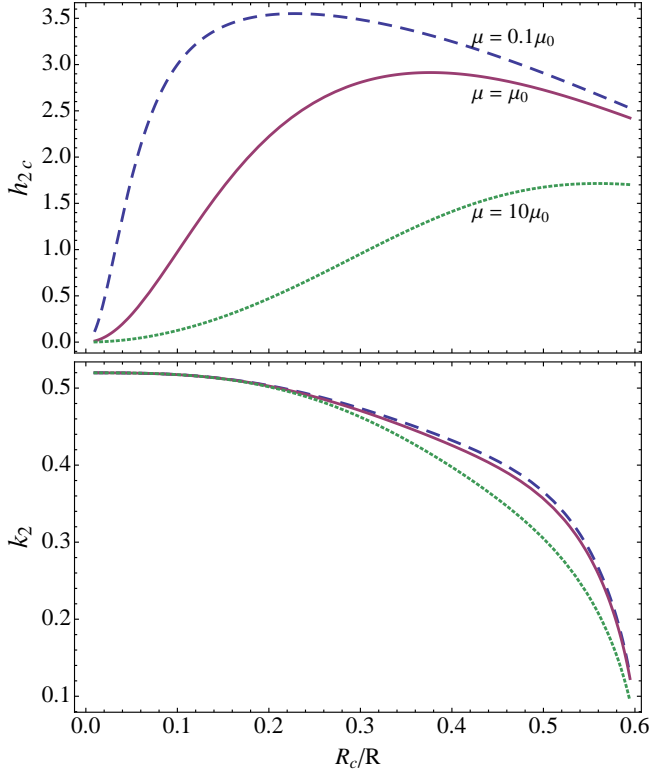


Figure 1. Sample tidal Love numbers h_{2c} (top) and k_2 (bottom) for a gas giant with mass M_J , radius R_J , and core density $\rho_c = 6 \text{ g cm}^{-3}$, as a function of core radius for several values of core shear modulus, where $\mu_0 = 900 \text{ kbar}$.

hardly changes the surface tidal k_2 (bottom panel). Core radius plays a slightly more important role, with smaller cores yielding smaller k_2 , as expected (note this is opposite to Fig. 1 because here we are keeping core mass rather than core density constant). Perhaps surprisingly, smaller cores are deformed more than larger cores (top panel). This can be understood by noting that the amount of deformation depends on $\bar{\mu}$, the ratio of the core shear modulus to the the gravitational rigidity $\rho_c g_c R_c$ of the core, with larger ratios yielding smaller deformations (cf. Eq. 30). For constant core mass, this ratio scales as $\bar{\mu} \propto R_c^4$, and therefore the core deforms less at higher radii.

4.2 Dissipation of viscoelastic core

We now consider tidal dissipation of a viscoelastic core, modeled by a complex shear modulus. We take

$$\mu \rightarrow \tilde{\mu} \equiv \tilde{\mu}_1 + i\tilde{\mu}_2, \quad (31)$$

and assume the simplest viscoelastic model - the Maxwell model, such that (Henning et al. 2009)

$$\tilde{\mu}_1 = \frac{\mu (\omega/\omega_M)^2}{1 + (\omega/\omega_M)^2}, \quad (32)$$

$$\tilde{\mu}_2 = -\frac{\mu (\omega/\omega_M)}{1 + (\omega/\omega_M)^2}, \quad (33)$$

where ω is the forcing frequency in the reference frame of the planet, and ω_M is the Maxwell frequency given by $\omega_M \equiv$

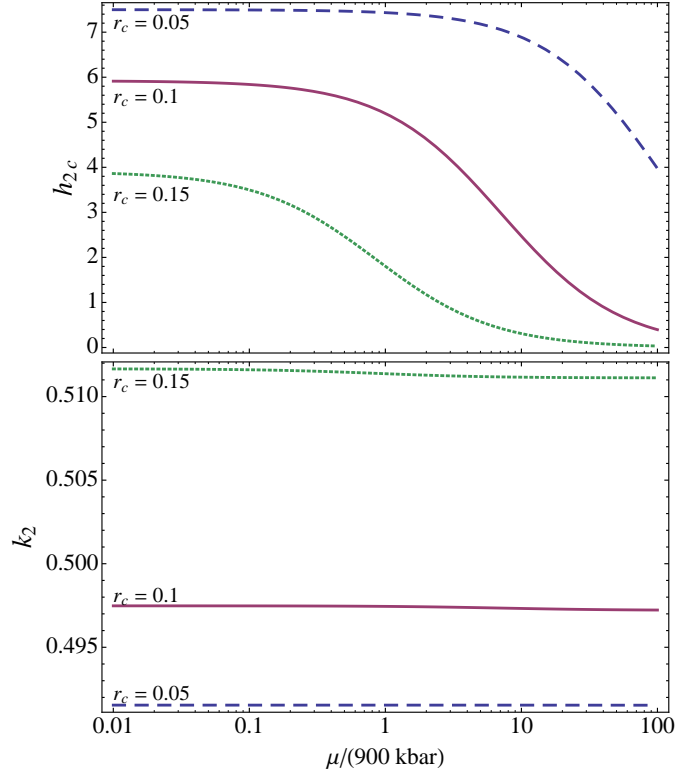


Figure 2. Sample tidal Love numbers h_{2c} (top) and k_2 (bottom) for a gas giant with mass M_J , radius R_J , and core mass $M_c = 5M_\oplus$, as a function of shear modulus μ for several values of fractional core radius $r_c \equiv R_c/R$.

μ/η , where μ is the normal shear modulus of the core and η the viscosity of the core. Under the Maxwell model, the solid core responds viscously for $\omega \lesssim \omega_M$ and elastically for $\omega \gtrsim \omega_M$.

Figure 3 shows tidal dissipation rates, characterized by $\text{Im}[k_2]$, as a function of the forcing frequency, for different values of μ and η . Since η only enters into the expression for μ through the ratio ω/ω_M , it is not surprising that changing η simply shifts the tidal dissipation curve horizontally without changing the strength (Fig. 3, bottom). The effect of μ is more complicated (Fig. 3, top panel) and can shift the curve up/down as well. Since μ does not directly affect the viscous properties of the core, it makes sense that a change in μ shifts the curve such that the tidal response on the viscous side ($\omega \lesssim \omega_M$) remains unchanged.

4.3 Comparison with planet models with uniform-density envelope

An analytical formula for the tidal number k_2 was previously derived for giant planet models with uniform envelope density (Dermott 1979). Recent works have used viscoelastic dissipation in the solid cores of such models to explain the amount of tidal dissipation inferred from the evolution of Jupiter's and Saturn's satellites (Remus et al. 2012, Remus et al. 2015) and from constraints on high-eccentricity migration of hot Jupiters (Storch & Lai 2014). In Figure 4 we compare the dissipation levels in planets with $n = 1$ vs uniform-density envelopes. While we do not attempt to ex-

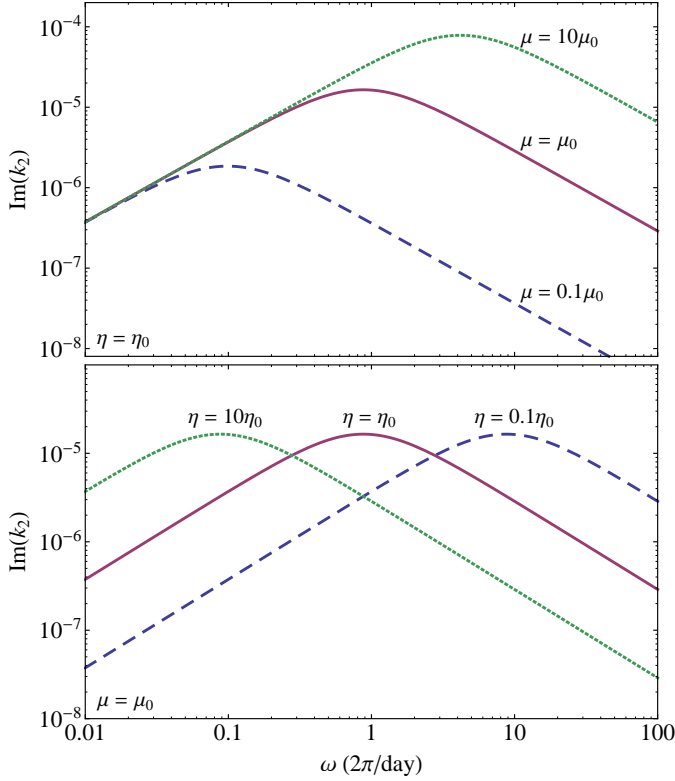


Figure 3. Sample tidal dissipation rates for a gas giant with mass M_J , radius R_J with an $n = 1$ envelope, characterized by $\text{Im}[k_2]$, as a function of the forcing frequency ω . Here $R_c = 0.1R$, and $M_c = 5M_\oplus$. Top panel: for a fixed viscosity $\eta = \eta_0 \approx 12.3 \text{ Gbar} \cdot \text{s}$ and three values of μ , were $\mu_0 = 900 \text{ kbar}$. Bottom panel: for a fixed $\mu = \mu_0$ and three values of η .

plore the full parameter space here, Figure 4 suggests that the difference between the two depends most strongly on the density of the core, with the $n = 1$ dissipation being stronger in more compact cores by as much as a factor of few.

4.4 Application to Super-Earths

While in this work we focus mainly on gas giants, the formulae presented in section 3 do not assume the core radius to be small. Thus, in principle, they may be applied to super-Earths - with the caveat that super-Earths are not likely to be well-described by $n = 1$ envelopes. In Figure 5 we present the Love numbers for a super-Earth analogue similar to Kepler-11d (Lissauer et al. 2013), consisting of a solid core and a $n = 1$ gas envelope that is $\sim 15\%$ by mass, but $\sim 50\%$ by radius. In this case the effect of the “core” shear modulus on the surface k_2 is significant when μ is varied by a few orders of magnitude.

5 CONCLUSION

We have presented a general method of computing the tidal Love numbers of giant planets consisting of a uniform elastic solid core and a non-uniform fluid envelope. We show that if the envelope obeys the $n = 1$ polytropic equation of state

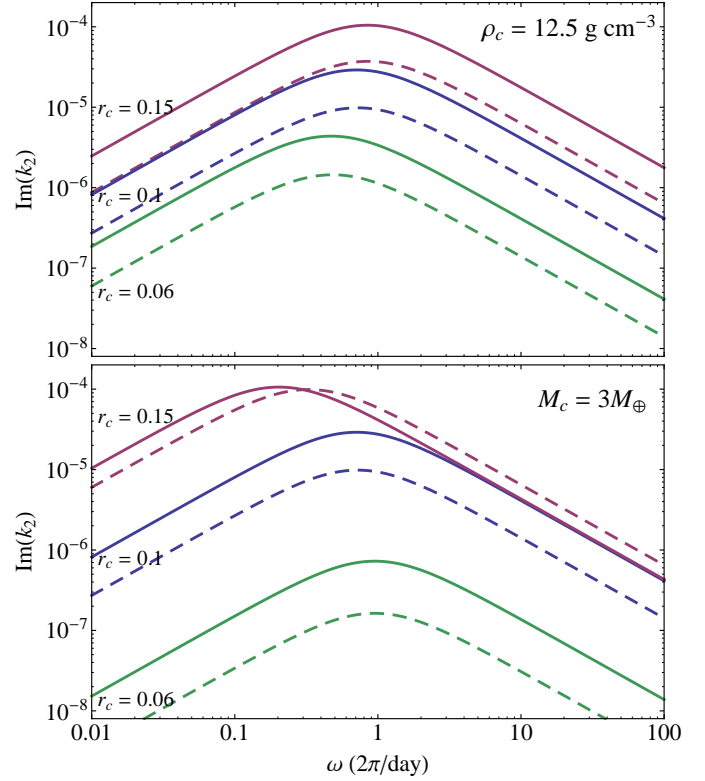


Figure 4. Comparison of tidal response curves for a gas giant with mass M_J , radius R_J with an $n = 1$ envelope (solid lines) vs $n = 0$ envelope (dashed lines). Top panel: for a solid core of fixed density $\rho_c = 12.5 \text{ g cm}^{-3}$ but varying radii $r_c \equiv R_c/R$. Bottom panel: for a solid core of fixed mass $M_c = 3M_\oplus$ but varying radii. The core mass and density in each panel have been selected such that the $r = 0.1$ curves are identical in the top and bottom panels. We assume $\mu = \mu_0 = 900 \text{ kbar}$ and $\eta = \eta_0 = 12 \text{ Gbar} \cdot \text{s}$.

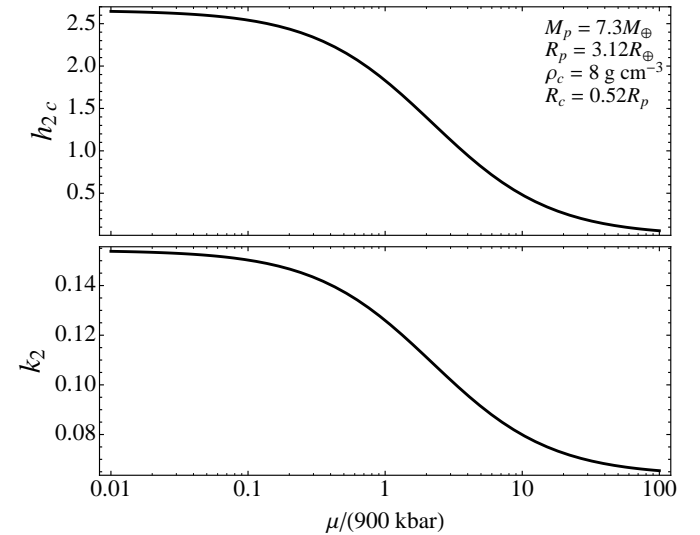


Figure 5. Tidal response for a super-Earth analogue similar to Kepler-11d, assuming an $n = 1$ envelope.

($P \propto \rho^2$), simple analytical expressions for the tidal Love numbers can be obtained [see Eqs. (23)-(26)]. These expressions are valid for any core size, density, and shear modulus. They allow us to compute the tidal dissipation rate in the viscoelastic core of a planet by using a complex shear modulus that characterizes the rheology of the solid core. Our results improve upon previous works that are based on planetary models with a solid core and a uniform ($n = 0$) envelope. In general, we find that while for diffuse (low density, larger size) cores, the dissipation rates of the $n = 0$ envelope models can be higher than the $n = 1$ models, for more compact cores the $n = 1$ envelope models have higher dissipation rates by as much as a factor of few.

While we have focused on analytical expressions for the Love numbers in this paper, our method and equations can be adapted for numerical computation of the real (non-dissipative) Love numbers for any envelope equation of state. In particular, they can be used to study tidal distortion of super-earths, which typically contain a H-He envelope (a few percent by mass) surrounding a rocky core (e.g., Lissauer, Dawson & Tremaine 2014), or tidal distortion in gas giants with more realistic equations of state. It is possible that for some exoplanetary systems, particularly those containing hot Jupiters, the tidal Love number k_2 can be constrained or measured using secular planet-planet interactions (e.g. Mardling 2010, Batygin & Becker 2013). This would provide a useful probe of the interior structure of the planet.

In the presence of viscosity in the solid core, additional work is still needed to obtain the tidal viscoelastic dissipation rate even when numerical results for the real Love numbers are available. Thus, our analytical expressions will be useful, as they allow for simple computation of the tidal dissipation rate via the correspondence principle, and can serve as a calibration of numerical results.

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